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SLIPPING REGIMES IN MECHANICAL SYSTEMS*

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Mechanical systems with non-bilateral kinematic constraints are considered. For such systems conditions are obtained for which their equations of motion, determined by methods from classical mechanics, convex underdetermination and equivalent control, are identical.

Problems associated with methods of obtaining the equations of motion in slipping regimes, appearing in systems of differential equations with discontinuous right-hand sides, have been most thoroughly discussed in /1, 2/. From the point of view of classical mechanics, the appearance of such regimes amounts to imposing on a system of material points $P_i, (i = 1, 2, \ldots, N)$ some non-bilateral relations S_β $(\beta = 1, 2, \ldots, m)$

$$S_\beta = \varphi_\beta (t, r, r^\cdot), \; r = (r_1, \ldots, r_N), \; r^\cdot = (r_1^\cdot, \ldots, r_N^\cdot) \tag{0.1}$$

where r_i and $r_i^\cdot$ are respectively the position vectors and velocities of the points P_i.

In this context slipping motion corresponds to motion in which the constraints S_β become bilateral during certain modes of behaviour, i.e. $S_\beta = 0$. Then the right-hand sides of the dynamical equations for the points P_i undergo discontinuities on the hypersurfaces $S_\beta = 0$.

Below we shall assume that when the links are bilateral (embedding), they are ideal. This assumption is natural for a wide class of mechanical systems.

For example, such a situation occurs when S_β is a non-bilateral frictional constraint /3/. If the constraint (0.1) is ideal (in the above sense) and linear in the velocities, then to derive the equations of motion for the points P_i of a constrained system one can apply Lagrange's method of undetermined multipliers (alternatively, the method of convex underdetermination and equivalent control /1, 2/). However, any one of these approaches in isolation may not give sufficient information for investigating the behaviour of such systems with variable structure. In particular, the Lagrange method, uniquely defining the slipping equations, does not, in general, establish their switching conditions, whereas the methods in /1, 2/, in principle giving conditions for the existence of singular regimes, in a range of cases do not guarantee the correctness of the derivation of the equations of motion when the constraints S_β are bilateral.

In connection with these and other problems there is the interesting problem of the consistency of the various methods of deriving equations of motions for slipping regimes for mechanical systems of variable structure within the framework of Newtonian mechanics.

1. We will first consider a dynamical system of the form

$$m_i x_{ij}^{\cdot\cdot} = F_{ij}(t, x, x^\cdot) + \sum_{\beta=1}^{m} b_{i\beta}^{j}(t, x, x^\cdot)\, u_\beta(x, x^\cdot) \tag{1.1}$$

$$u_\beta = \begin{cases} u_\beta^+, S_\beta > 0, \\ u_\beta^-, S_\beta < 0, \end{cases} \quad S_\beta = \sum_{i, \;=1}^{N, 3} l_{\beta i} x_{ij}^\cdot + l_{\beta(N+1)}^4 \tag{1.2}$$

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$$l_{\beta i}{}^j = l_{\beta i}{}^j(x),\ l^4_{\beta(N+1)} = l^4_{\beta(N+1)}(x);\ i = 1, 2, \ldots, N;\ j = 1, 2, 3$$
$$x = (x_1, \ldots, x_N), \quad x^{\cdot} = (x_1^{\cdot}, \ldots, x_N^{\cdot})$$
$$x_i = (x_{i1}, x_{i2}, x_{i3}), \quad x_i^{\cdot} = (x_{i1}^{\cdot}, x_{i2}^{\cdot}, x_{i3}^{\cdot})$$

Here m_i is the mass of the i-th point and x_i and $x_i^{\cdot}$ are its three-dimensional coordinates and velocity.

The differential constraint S_β ($\beta = 1, 2, \ldots, m$), considered as a scalar function of variables x and $x^{\cdot}$, is a surface of discontinuity for $u_\beta(x, x^{\cdot})$. If $S_\beta \neq 0$, $\forall\beta = 1, 2, \ldots, m$, then the right-hand sides of system (1.1) are active forces (alongside F_{ij}). If one of the constraints S_r ($r = 1, 2, \ldots, m$) is included, then the corresponding $b_{ir}{}^j u_r$ terms determine the reaction of the constraint S_r.

Below we shall use the following notation:

$$F_{ij}^r[\varphi^{\cdot}(u)] = F_{ij} + \sum_{\beta=1,\ \beta\neq r}^{m} b_{i\beta}^j \varphi_{i\beta}^j(u_\beta) \tag{1.3}$$
$$\alpha_r[\varphi(u)] = \sum_{\nu,\ k=1}^{N,\ 3} \frac{F_{ij}{}^r[\varphi_{i\beta}{}^j(u_\beta)]\, l_{r\nu}^k}{m_\nu} + \Gamma_r$$
$$a_r(\varkappa, \omega, \sigma, \tau) = \sum_{\nu,\ k=1}^{N,\ 3} \frac{\varkappa_{r\nu}^k \omega_{r\nu}^k \sigma_{\nu r}^k \tau_{\nu r}^k}{m_\nu}$$
$$\Gamma_r = \sum_{\nu,\ k=1}^{N,\ 3} \langle \nabla l_{r\nu}^k x^{\cdot}\rangle\, x_{\nu k}^{\cdot} + \langle \nabla l_{r(N+1)}^4 x^{\cdot}\rangle$$

(where $\langle\cdot\rangle$ denotes the scalar product).

We shall write the equations of motion for the system with an imposed constraint S_r' using Lagrange's method /4/ and the method of equivalent control /2/, which for system (1.1) leads to the same result as Filippov's method /1/:

$$m_i x_{ij}^{\cdot\cdot} = F_{ij}{}^r(u) + l_{ri}{}^j \lambda_r \tag{1.4}$$
$$m_i x_{ij}^{\cdot\cdot} = F_{ij}{}^r(u) + b_{ir}{}^j u_r{}^{eq} \tag{1.5}$$

Here

$$\lambda_r = -\frac{\alpha_r(u)}{a_r(l, l, 1, 1)}, \quad u_r^{eq} = -\frac{\alpha_r(u)}{a_r(l, 1, b, 1)} \tag{1.6}$$
$$\min(u_r^+, u_r^-) \leqslant u_r{}^{eq} \leqslant \max(u_r^+, u_r^-) \tag{1.7}$$

The functions $F_{ij}{}^r(u), \alpha_r(u)$ and a_r are given by (1.3) with $\varphi(u) = u$.

We note that it follows from relations (1.6) that λ_r and $u_r{}^{eq}$ are both either zero or non-zero. We shall not consider the trivial case when Eqs.(1.4) and (1.5) as identical when $\lambda_r = u_r{}^{eq} = 0$.

It turns out that for Eqs.(1.4) and (1.5) to be identical it is necessary and sufficient for the system of equations

$$l_{ri}{}^j b_{\nu r}{}^k = b_{ir}{}^j l_{r\nu}{}^k, \quad \forall i, \nu = 1, 2, \ldots, N; \quad j, k = 1, 2, 3 \tag{1.8}$$

to be satisfied.

Proof. Necessity. Suppose Eqs.(1.4) and (1.5) are identical. Then λ_r and $u_r{}^{eq}$ will satisfy the linear system of equations

$$l_{ri}{}^j \lambda_r - b_{ir}{}^j u_r{}^{eq} = 0 \tag{1.9}$$

Because λ_r and $u_r{}^{eq}$ are non-zero by assumption, the rank of system (1.9) is less than two, from which we obtain condition (1.8).

Sufficiency. Suppose relation (1.8) is satisfied. Taking any non-zero $l_{ri}{}^j$ and $b_{ir}{}^j$ and expressing all the $b_{\nu r}{}^k$ in (1.8) in terms of the remaining quantities, we substitute into the second equations of (1.6). We obtain

$$u_r{}^{eq} = (l_{ri}{}^j / b_{ir}{}^j)\, \lambda_r$$

This last relation leads to the equivalence of the right-hand sides of (1.4) and (1.5). Here and below it is assumed that F_{ij} and $b_{i\beta}{}^j$ are continuous in x and x', and Lebesgue-meansurable in t, while $l_{\beta i}{}^j$ and $l_{\beta(N+1)}^1$ are continuously differentiable in all their arguments.

Remark. If some constraints $S_1, \ldots, S_z$, $z \leqslant m, m \neq 3N$ are imposed on system (1.1), then in this case conditions (1.8) are also sufficient for the equivalence of the equations of motion obtained by these three methods. In order for motion to exist at the intersections of the discontinuity surfaces $S_1, \ldots, S_z$, it is necessary (even for existence along each of them), for conditions (1.12) to be equivalence criteria for equations of slipping motion obtained by the various methods. For $z = m, m = 3N$, the equations of slipping motion are the same for all $b_{i\beta}^j$ because in this case they are uniquely defined by the constraint equations.

2. We consider the case of non-linear activation of discontinuous functions on the right-hand sides of the dynamic equations

$$m_i x_{ij}^{\cdot\cdot} = F_{ij}(t, x, x^\cdot) + \sum_{\beta=1}^{m} b_{i\beta}^j(t, x, x^\cdot)\,\varphi_{i\beta}^j(u_\beta) \tag{2.1}$$

with imposed constraints (1.2). The functions $\varphi_{i\beta}^j(u_\beta)$ are continuous in u_β. The equations of motion obtained using the Lagrange method with the constraint S_β switched on have the form

$$m_i x_{ij}^{\cdot\cdot} = F_{ij}^{\cdot\cdot}[\varphi(u)] - l_{ri}^j \alpha_r[\varphi(u)]/a_r(l,\ l,\ 1,\ 1) \tag{2.2}$$

It was noted in /2/ that the results of applying the methods of convex underdetermination and equivalent control to systems of the form (2.1) do not, in general, coincide, and to ensure the correctness of the derivation of the slipping equations additional information is necessary on the nature of the object under investigation. Eqs.(2.2) can provide this information for the case under consideration.

We will compare (2.2) with the system obtained by the method of convex underdetermination. Applying the standard Filippov technique, after reduction we obtain

$$m_i x_{ij}^{\cdot\cdot} = F_{ij}^r[\varphi(u)] - \frac{\Delta\varphi_{ir}^j b_{ir}^j \alpha_r[\varphi(u)] + \gamma_{ir}^j}{a_r(l, 1, b, \Delta\varphi)} \tag{2.3}$$

$$\Delta\varphi_{ir}^j = \varphi_{ir}^j(u_r^+) - \varphi_{ir}^j(u_r^-)$$

$$\gamma_{ir}^j = \sum_{\nu, k=1}^{N, 3} \frac{l_{r\nu}^k b_{\nu i}^k b_{ir}^j}{m_\nu}\,\Phi_{\nu ir}^{kj}$$

$$\Phi_{\nu ir}^{kj} = \varphi_{\nu r}^k(u_r^-)\,\varphi_{ir}^j(u_r^+) - \varphi_{\nu r}^k(u_r^+)\,\varphi_{ir}^j(u_r^-)$$

Here it is assumed that $\Delta\varphi_{ir}^j \neq 0$. Comparing the right-hand sides of systems (2.2) and (2.3) with the arbitrary active forces F_{ij}, we obtain the system

$$l_{ri}^j a_r(l,\ 1,\ b,\ \Delta\varphi) - b_{ir}^j \Delta\varphi_{ir}^j a_r(l,\ l,\ 1,\ 1) = 0 \tag{2.4}$$

$$i = 1, 2, \ldots, N;\quad j = 1, 2, 3;\quad r = 1, 2, \ldots, m$$

Assuming that $a_r(l, 1, b, \Delta\varphi) \neq 0$ and $a_r(l, l, 1, 1) \neq 0$, we obtain as before the relations

$$l_{ri}^j \Delta\varphi_{\nu r}^k b_{\nu r}^k - l_{r\nu}^k \Delta\varphi_{ir}^j b_{ir}^j = 0 \tag{2.5}$$

We further equate the right-hand sides of systems (2.2) and (2.3) and substitute into this equality for the quantity $b_{\nu r}^k$ using the remaining quantities in expression (2.5). With the natural requirement that l_{ri}^j and φ_{ri}^j do not depend on the mass parameters, we finally obtain

$$\Phi_{\nu ir}^{kj} = 0 \tag{2.6}$$

Thus (2.5) and (2.6) are the conditions for the equivalence of the slipping equations obtained by the Lagrange and convex underdetermination methods.

Conditions for the equivalence of slipping equations obtained by the three specified methods can be written in the form

$$\frac{l_{ri}^j}{l_{r\nu}^k} = \frac{b_{ir}^j \Delta\varphi_{ir}^j}{b_{\nu r}^k \Delta\varphi_{\nu r}^k} = \frac{b_{ir}^j \varphi_{ir}^j(u_r^\pm)}{b_{\nu r}^k \varphi_{\nu r}^k(u_r^\pm)} = \frac{b_{ir}^j \varphi_{ir}^j(u_r^{eq})}{b_{\nu r}^k \varphi_{\nu r}^k(u_r^{eq})} \tag{2.7}$$

where the u_r^{eq} are determined as in /2/.

Example.

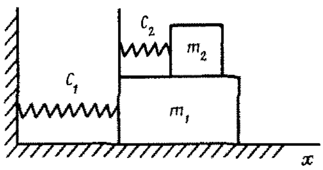
$$m_\beta x_{\beta 1}^{\cdot\cdot} = F_{\beta 1}(x_{11}, x_{21}) + b_{\beta 1}^1 \varphi_{\beta 1}^1(u_1) + b_{\beta 2}^1 \varphi_{\beta 2}^1(u_2) \tag{2.8}$$

$$u_\beta = \begin{cases} 1, & S_\beta > 0, \\ -1, & S_\beta < 0, \end{cases} \quad S_\beta = l_{\beta 1}^1 x_{11}^\cdot + l_{\beta 2}^1 x_{21}^\cdot, \quad \beta = 1, 2$$

If (2.8) is interpreted as the mechanical system with "dry" friction shown in the figure, then

$$F_{11} = -c_1x_{11} - c_2(x_{11} - x_{21}),\ F_{21} = c_2(x_{11} - x_{21}) \tag{2.9}$$
$$b_{1\beta}{}^1 = -m_\beta g k_\beta,\ b_{21}{}^1 = 0,\ b_{22}{}^1 = m_2 g k_2$$
$$l_{\beta 1}{}^1 = 1,\ l_{12}{}^1 = 0,\ l_{22}{}^1 = -1.$$
$$\varphi^1_{\beta i}(u_\beta) = \begin{cases} 1, & S_\beta > 0, \\ -1, & S_\beta < 0 \end{cases} \quad \beta = 1, 2;\ i = 1, 2$$

i.e. $\varphi_{\beta i}{}^1$ can be any odd functions with limiting values identical with the limiting values of u_β. Here c_1 and c_2 are the respective rigidities of the springs, g is the acceleration due to gravity, and k_1 and k_2 are the frictional coefficients.



If $\varphi_{\beta i}{}^1(u_\beta) = u_\beta, \forall i,\ \beta = 1, 2,$ then the equations of motion obtained with the inclusion of any constraint S_β will be identical for all three methods, because condition (1.3) is satisfied.

Indeed,

$$l_{11}{}^1 b_{21}{}^1 = 0,\quad l_{12}{}^1 b_{11}{}^1 = 0,\quad \Rightarrow l_{11}{}^1 b_{21}{}^1 = l_{12}{}^1 b_{11}{}^1$$
$$b_{12}{}^1 l_{22}{}^1 = m_2 g k_2,\quad l_{21}{}^1 b_{22}{}^1 = m_2 g k_2,\quad \Rightarrow b_{12}{}^1 l_{22}{}^1 = b_{22}{}^1 l_{21}{}^1$$

If the $\varphi_{\beta i}{}^1(u_\beta)$ are non-linear in u_β and satisfy conditions (2.9), then the equations of the slipping regimes obtained by the Lagrange and Filippov methods will also be identical, because conditions (2.5) and (2.6) are satisfied:

$$\Delta\varphi_{\beta i}{}^1 = 2,\ l_{11}{}^1 2b_{21}{}^1 = 0,\ l_{12}{}^1 2b_{11}{}^1 = 0$$
$$l_{21}{}^1 2b_{22}{}^1 = l_{22}{}^1 2b_{12}{}^1 = 2m_2 g k_2,\ \beta,\ i = 1,\ 2$$

Consequently,

$$l^1_{ri}\Delta\varphi^1_{\nu r} b^1_{\nu r} = l^1_{r\nu}\Delta\varphi^1_{ir} b^1_{ir},\quad \forall r, i, \nu = 1, 2$$

The satisfaction of this condition follows directly from (2.9). As an example we will write out the equations of motion with the inclusion of the constraint $S_2 = 0$, i.e. when the relative velocity of the masses m_1 and m_2 is zero:

$$(m_1 + m_2)\, x_{11}^{\cdot\cdot} = F_{11} + F_{21} - m_1 g k_1 \varphi_{11}{}^1(u_1) \tag{2.10}$$

with conditions

$$|\, G/[m_2(m_1 + m_2)\, g k_2]\,| \leqslant 1$$
$$G = F_{11} m_2 - F_{21} m_1 - m_1 m_2 g k_1 \varphi_{11}{}^1(u_1)$$

The method of equivalent control can lead to different dynamical equations when the constraint $S_2 = 0$ is imposed and in the case of a non-linear dependence of $\varphi_{2i}{}^1$ on u_2 satisfying conditions (2.9). Suppose $\varphi_{21}{}^1(u_2) = u_2$ and $\varphi_{22}{}^1(u_2) = (u_2)^3$. Then the equivalent control $u_2{}^{eq}$ will satisfy the equation

$$g k_2 (u_2{}^{eq})^3 + (m_2/m_1)\, g k_2 u_2{}^{eq} + G/(m_1 m_2) = 0 \tag{2.11}$$

On the other hand, according to (2.7) the quantity $u_2{}^{eq}$ should satisfy the relation

$$\frac{l_{21}{}^1}{l_{22}{}^1} = \frac{b_{12}{}^1 u_2^{eq}}{b_{22}{}^1 (u_2^{eq})^3},\quad \Rightarrow u_2^{eq} = \pm 1$$

Clearly, neither of these values for $u_2{}^{eq}$ will satisfy Eq.(2.11) if the force characteristics of the system are not formally constrained by special relations characterizing the unstable state of the system.

3. We shall show that conditions of type (1.8) also hold in the case when a mechanical system in which discontinuous functions occur linearly is written in generalized coordinates $q_1, q_2, \ldots, q_n$.

Suppose we have a mechanical system characterized by kinematic and potential energies T and V and acted upon by generalized forces

$$Q_i = F_i(t, q, q^{\cdot}) + \sum_{\beta=1}^{m} b_{i\beta}(t, q, q^{\cdot})\, u_\beta$$
$$T = T_1 + T_2 = \sum_{i=1}^{n} a_i(q)\, q_i^{\cdot} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)\, q_i^{\cdot} q_j^{\cdot}$$

$$q = (q_1, \ldots, q_n), \quad q^{\cdot} = (q_1^{\cdot}, \ldots, q_n^{\cdot})$$

$$u_\beta = \begin{cases} u_\beta^+, S_\beta > 0, \\ u_\beta^-, S_\beta < 0, \end{cases} S_\beta = \sum_{i=1}^n B_{\beta i}(q) q_i^{\cdot} + B_\beta(q)$$

$$\beta = 1, 2, \ldots, m$$

Then, following /5, 2/, we conclude that when some constraint $S_r = 0$ is included the equations of the slipping regime will have the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{\cdot}}\right) - \frac{\partial L}{\partial q_i} = F_i^r + B_{ri}\lambda_r \tag{3.1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{\cdot}}\right) - \frac{\partial L}{\partial q_i} = F_i^r + b_{ir}u_r^{eq}$$

$$F_i^r = F_i + \sum_{\beta=1, \beta\neq r}^m b_{i\beta}u_\beta, \quad L = T - V \tag{3.2}$$

λ_r and u_r^{eq} are found from (3.1) and (3.2) together with the equation $S_r' = 0$. As above, it can be shown that for Eqs.(3.1) and (3.2) to be identical the conditions

$$B_{ri}b_{jr} = B_{rj}b_{ir}, \quad \forall i, \ j = 1, 2, \ldots, n \tag{3.3}$$

are necessary.

We shall show their sufficiency. Write systems (3.1) and (3.2) in the expanded form

$$\sum_{j=1}^n a_{ij}q_j^{\cdot\cdot} = P_i + B_{ri}\lambda_r \tag{3.4}$$

$$\sum_{j=1}^n a_{ij}q_j^{\cdot\cdot} = P_i + b_{ir}u_r^{eq} \tag{3.5}$$

$$\sum_{i=1}^n B_{ri}q_i^{\cdot} + B_r = 0 \tag{3.6}$$

where the P_i depend only on q, $q^{\cdot}$ and t.

From (3.4), (3.6) and (3.5), (3.6) we find

$$\lambda_r = -(D^r\Delta + \Sigma_r)\left(\sum_{\nu, j=1}^n B_{r\nu}B_{rj}A_{j\nu}\right)^{-1} \tag{3.7}$$

$$u_r^{eq} = -(D^r\Delta + \Sigma_r)\left(\sum_{\nu, j=1}^n B_{r\nu}b_{jr}A_{j\nu}\right)^{-1} \tag{3.8}$$

$$D^r = \sum_{\nu=1}^n \langle \nabla B_{r\nu} q^{\cdot}\rangle q_\nu^{\cdot} + \langle \nabla B_r q^{\cdot}\rangle$$

$$\Sigma_r = \sum_{\nu, j=1}^n B_{r\nu}P_jA_{j\nu}$$

$$\Delta = \det \| a_{ij} \|_{j=1, \ldots, n}^{i=1, \ldots, n}$$

where the A_{ji} are the cofactors of the elements a_{ij}.

Using relations (3.3) we express all the b_{jr} in terms of the remaining quantities and substitute into Eqs.(3.8). We obtain the result

$$u_r^{eq} = (B_{ri}/b_{ir})\lambda_r$$

which leads to the equivalence of Eqs.(3.1) and (3.2).

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